Neutrino Masses and Mixings in a Minimal SO(10) Model

K.S. Babu[*]

Department of Physics, Oklahoma Center for High Energy Physics,

Oklahoma State University, Stillwater, OK 74078, USA

C. Macesanu[†]

Department of Physics, Oklahoma Center for High Energy Physics,

Oklahoma State University, Stillwater, OK 74078, USA and

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA

We consider a minimal formulation of $SO(10)$ Grand Unified Theory wherein all the fermion masses arise from Yukawa couplings involving one $\overline{\mathbf{126}}$ and one $\mathbf{10}$ of Higgs multiplets. It has recently been recognized that such theories can explain, via the type–II seesaw mechanism, the large $\nu_\mu - \nu_\tau$ mixing as a consequence of $b-\tau$ unification at the GUT scale. In this picture, however, the CKM phase δ lies preferentially in the second quadrant, in contradiction with experimental measurements. We revisit this minimal model and show that the conventional type–I seesaw mechanism generates phenomenologically viable neutrino masses and mixings, while being consistent with CKM CP violation. We also present improved fits in the type–II seesaw scenario and suggest fully consistent fits in a mixed scenario.

I. INTRODUCTION

Grand Unified Theories (GUT) provide a natural framework to understand the properties of fundamental particles such as their charges and masses. GUT models based on $SO(10)$ gauge symmetry have a number of particularly appealing features. All the fermions in a family fit in a single 16–dimensional spinor multiplet of $SO(10)$. In order to complete this multiplet, a right–handed neutrino field is required, which would pave the way for the seesaw mechanism which explains the smallness of left–handed neutrino masses. $SO(10)$ contains $SU(5)$ and the left–right symmetric Pati–Salam symmetry group $SU(4)_C \times SU(2)_L \times SU(2)_R$ as subgroups, both with

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[*]Electronic address: babu@okstate.edu

[†]Electronic address: cmacesan@physics.syr.edu

very interesting properties from a phenomenological perspective. With low energy supersymmetry, $SO(10)$ and $SU(5)$ models also lead remarkably to the unification of the three Standard Model gauge couplings at a scale $M_{\mathrm{GUT}} \sim 2 \times 10^{16}$ GeV.

In grand unified theories, the gauge sector and the fermionic matter sector are generally quite simple. However, the same is not true of the Higgs sector. Since the larger symmetry needs to be broken down to the Standard Model, generally one needs to introduce a large number of Higgs multiplets, with different symmetry properties under gauge transformations. If all of these Higgs fields couple to the fermion sector, one would lose much of the predictive power of the theory in the masses and mixings of quarks and leptons, and so also one of the attractive aspects of GUTs.

Of interest then are the so–called minimal unification theories, in which only a small number of Higgs multiplets couple to the fermionic sector. One such realization is the minimal $SO(10)$ GUT [1] in which only one **10** and one $\overline{\mathbf{126}}$ of Higgs fields couple to the fermions. These two Higgs fields are responsible for giving masses to all the fermions of the theory, including large Majorana masses to the right–handed neutrinos. This model is minimal in the following sense. The fermions belong to the **16** of $SO(10)$, and the fermion bilinears are given by $\mathbf{16} \times \mathbf{16} = \mathbf{10}_s + \mathbf{120}_a + \mathbf{126}_s$. Thus **10**, **120** and $\overline{\mathbf{126}}$ Higgs fields can have renormalizable Yukawa couplings. If only one of these Higgs fields is employed, there would be no family mixings, so two is the minimal set. The $\overline{\mathbf{126}}$ has certain advantages. It contains a Standard Model singlet field and so can break $SO(10)$ down to $SU(5)$, changing the rank of the group. Its Yukawa couplings to the fermions also provide large Majorana masses to the right–handed neutrinos leading to the seesaw mechanism. It was noted in Ref. [1] that due to the cross couplings between the $\overline{\mathbf{126}}$ and the **10** Higgs fields, the Standard Model doublet fields contained in the $\overline{\mathbf{126}}$ will acquire vacuum expectation values (VEVs) along with the VEVs of the Higgs doublets from the **10**. The $\overline{\mathbf{126}}$ Yukawa coupling matrix will then contribute both to the Dirac masses of quarks and leptons, as well as to the Majorana masses of the right–handed neutrinos.

It is not difficult to realize that this minimal model is highly constrained in explaining the fermion masses and mixings. There are two complex symmetric Yukawa coupling matrices, one of which can be taken to be real and diagonal without loss of generality. These matrices have 9 real parameters and six phases. The mass matrices also depend on two ratios of VEVs, leading to 11 magnitudes and six phases in total in the quark and lepton sector, to be compared with the 13 observables (9 masses, 3 CKM mixings and one CP phase). Since the phases are constrained to be between -1 and $+1$, this system does provide restrictions. More importantly, once a fit is found for the charged fermions, the neutrino sector is fixed in this model. It is not obvious at all that

including the neutrino sector the model can be phenomenologically viable.

Early analyses [1, 2] found that just fitting the lepton-quark sector is highly constraining. Also, this fitting has been found to be highly nontrivial (in terms of complexity); therefore these analyses were done in the limit when the phases involved are either zero or π. In such a framework, one finds that the parameters of the models are more or less determined by the fit to the lepton-quark sector (the quark masses themselves are not known with great precision, so there is still some room for small variations of the parameters). As a consequence, one could more or less predict the neutrino masses and mixings; however, since neutrino data was rather scarce at the time, one could not impose meaningful constraints on the minimal $SO(10)$ model from these predictions.

In view of the new information on the neutrino sector gathered in the past few years [3, 4, 5], one should ask if this model is still consistent with experimental data. Interest in the study of this model has also been reawakened by the observation that $b - \tau$ unification at GUT scale implies large (even close to maximal) mixing in the 2-3 sector of the neutrino mass matrix [6], provided that the dominant contribution to the neutrino mass is from type–II seesaw. There has been a number of recent papers studying the minimal $SO(10)$ using varying approaches: some analytical, concentrating on the 23 neutrino sector [6, 7], some numerical, either in the approximation that the phases involved in reconstructing the lepton sector are zero [8, 9, 10, 11], or taking these phases into account [12, 13]. The conclusions of these analyses seem to be that the minimal $SO(10)$ cannot account by itself for the observed neutrino sector (although it comes pretty close). However, one might restore agreement with the neutrino data if one slightly modifies the minimal $SO(10)$; for example, one can set the quark sector CKM phase to lie in the second quadrant, and rely on new contributions from the SUSY breaking sector in order to explain data on quark CP violation [12]; or one might add higher dimensional operators to the theory [12, 13], or even another Higgs multiplet (a **120**) which will serve as small perturbation to fermion masses [14, 15, 16].

In this paper we propose to revisit the analysis for the *minimal $SO(10)$* model, with no extra fields added. The argument for this endeavor is that our approach is different in two significant ways from previous analyses. First, we use a different method than [12, 13] in fitting for the lepton–quark sector. Since this fit is technically rather difficult, and moreover, since the results of this fit define the parameter space in which one can search for an acceptable prediction for the neutrino sector, we think that it is important to have an alternative approach. Second, rather than relying on precomputed values of quark sector parameters at GUT scale, we use as inputs M_Z scale values, and run them up to unification scale. This allows for more flexibility and we think more reliable predictions for the parameter values at GUT scale. With these modifications

in our approach, we find that we agree with some results obtained in [12, 13] (in particular, the fact that type–II seesaw does not work well when the CKM phase is in the first quadrant), but not with others. Most interesting, we find that it is possible to fit the neutrino sector in the minimal $SO(10)$ model, in the case when type–I seesaw contribution to neutrino mass dominates. We also present a mixed scenario which gives excellent agreement with the neutrino data.

The paper is organized as follows. In the next section we give a quick overview of the features of the minimal $SO(10)$ model relevant for our purpose. In section III we address the problem of fitting the lepton–quark sector in this framework. We also define the experimentally allowed range in which the input parameters (quark and lepton masses at M_Z scale) are allowed to vary. We start section IV with a quick overview of the phenomenological constraints on the neutrino sector. There we provide a very good fit to all the fermion masses and mixings using type–I seesaw. We follow by analyzing the predictions of the minimal $SO(10)$ model in the case when type–II seesaw is the dominant contribution to neutrino masses. We then analyze the predictions in a type–I seesaw dominance scenario, and in a scenario when both contributions (type–I and type–II) have roughly the same magnitude. We end with our conclusions in Sec. V.

II. THE MINIMAL SO(10) MODEL

The model we consider in this paper is an supersymmetric $SO(10)$ model where the masses of the fermions are given by coupling with only two Higgs multiplets: a **10** and a **$\overline{126}$** [1]. Both the $\mathbf{H_{10}}$ and $\mathbf{H_{\overline{126}}}$ contain Higgs multiplets which are **(2,2)** under the $SU(2)_L \times SU(2)_R$ subgroup. Most of these **(2,2)** Higgses acquire mass at the GUT scale. However, one pair of Higgs doublets H_u and H_d (which generally are linear combinations of the original ones) will stay light. (Details about the Higgs multiplet decomposition and $SO(10)$ breaking can be found, for example, in [17, 18, 19, 20]). Upon breaking of the $SU(2)_L \times U(1)_Y$ symmetry of the Standard Model, the vacuum expectation value of the H_u doublet will give mass to the up-type quarks and will generate a Dirac mass term for the neutrinos, while the vacuum expectation value of the H_d doublet will give mass to the down-type quarks and the charged leptons.

The mass matrices for quarks and leptons wil then have the following form:

$$M_u = \kappa_u Y_{10} + \kappa'_u Y_{126}$$
$$M_d = \kappa_d Y_{10} + \kappa'_d Y_{126}$$
$$M_\nu^D = \kappa_u Y_{10} - 3\kappa'_u Y_{126}$$

$$M_l = \kappa_d Y_{10} - 3\kappa'_d Y_{126} \tag{1}$$

where Y_{10}, Y_{126} are the Yukawa coefficients for the coupling of the fermions to the $\mathbf{H_{10}}$ and $\mathbf{H_{\overline{126}}}$ multiplets respectively. Note that in the above equations the parameters $\kappa_{u,d}, \kappa'_{u,d}$ as well as the Yukawa matrices are in general complex, thus insuring that the fermion mass matrices will contain CP violating phases.

The $SO(10)$ $\mathbf{H_{\overline{126}}}$ multiplet also contains a $(\overline{\mathbf{10}}, \mathbf{3}, \mathbf{1})$ and a $(\mathbf{10}, \mathbf{1}, \mathbf{3})$ Pati-Salam multiplets. The Higgs fields which are color singlets and $SU(2)_R/SU(2)_L$ triplets (denoted by Δ_R and Δ_L) may provide Majorana mass term for the right–handed and the left–handed neutrinos. One then has:

$$M_{\nu R} = \langle \Delta_R \rangle Y_{126}$$
$$M_{\nu L} = \langle \Delta_L \rangle Y_{126} . \tag{2}$$

If the vacuum expectation of the Δ_R triplet is around 10^{14} GeV then the Majorana mass term for the right–handed neutrinos will give rise, through the seesaw mechanism, to left–handed neutrino masses of order eV. On the other hand, the VEV of Δ_L contributes directly to the left–handed neutrino mass matrix (this contribution is called type–II seesaw), so this requires that the $\langle \Delta_L \rangle$ is either zero or at most of order eV. This requirement is satisfied naturally in such models, since Δ_L generally aquires a VEV of order $M_Z^2/\langle \Delta_R \rangle$ [21].

III. LEPTON AND QUARK MASSES AND MIXINGS

Our first task is to account for the observed lepton and quark masses, and for the measured values of the CKM matrix elements. By expressing the Yukawa matrices Y_{10} and Y_{126} in Eqs. (1) in favor of M_u and M_d, we get a linear relation between the lepton and quark mass matrices; at GUT scale:

$$M_l = a\, M_u + b\, M_d , \tag{3}$$

where a and b are a combinations of the $\kappa_{u,d}, \kappa'_{u,d}$ parameters in Eq. (1). For simplicity let's work in a basis where M_d is diagonal (this can be done without loss of generality). Then, $M_u = \mathcal{V}_{CKM}^T M_u^d \, \mathcal{V}_{CKM}$ with M_u^d the diagonal up–quark mass matrix. If we allow the entries in the diagonal quark mass matrices to be complex: $M_u^d = diag\{m_u e^{ia_u}, m_c e^{ia_c}, m_t e^{ia_t}\}$, $M_d^d = diag\{m_d e^{ib_d}, m_s e^{ib_s}, m_b e^{ib_b}\}$, then the CKM matrix can be written in its standard form as a

function of three real angles and a phase:

$$
\mathcal{V}_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} . \tag{4}
$$

Since their phases can be absorbed in the definitions of the a_i, b_i parameters, we will take the coefficients a, b to be real, too. One of the quark mass phases can be set to zero without loss of generality, we set $b_b = 0$. It should be noted that a common phase of $\{a,\ b\}$, which we denote as σ, will appear in the Dirac and Majorana mass matrices of the neutrinos, and will be relevant to the study of neutrino oscillations.

The relation (3) will generally impose some constraints on the masses of the quarks and leptons. For example, if we take all the phases to be zero (or π), then on the right-hand side of the equation there are just two unknowns, the coefficients a, b. On the other hand, the eigenvalues of the lepton mass matrix are known, which will give us 3 equations. It is not obvious, then, that this system can be solved; however, early analysis [1, 2] shows that solutions exist, in the range of experimentally allowed values, provided that the quark masses satisfy some constraints.

Newer studies [10, 11, 12] allow for (some) phases to be non-zero, and thus relaxes somewhat the constraints on quark masses. However, it is interesting to note that these solutions are not very different from the purely real case. That is, most of the phases involved have to be close to zero (or π), and the values of the parameters do not change by much. We shall explain this in the following.

The algebraic problem of solving for the lepton masses in the case when the elements of the matrices M_u, M_d are complex is quite difficult. This would involve solving a system of 3 polynomial equations of degree six in unknown quantities a, b. Most of the analysis so far is done by numerical simulations (some analytical results are obtained for the case of 2nd and 3rd families only [6, 7]). In this section we attempt to solve the full problem (with all the phases nonzero) in a semi-analytical manner, that is, by identifying the dominant terms in the equations and obtaining an approximate solution in the first step, which can then be made more accurate by successive iterations.

Due to the hierarchy between the eigenvalues of the lepton mass matrix $m_e \ll m_\mu \ll m_\tau$ one can suspect that the mass matrix itself has a hierarchical form. This assumption is supported by the observation that the off-diagonal elements of M_l are indeed hierarchical; for example $L_{13}/L_{23} \simeq \mathcal{V}_{31}/\mathcal{V}_{31} \ll 1$. ($L_{ij}$ is a short–hand notation for $(M_l)_{ij}$, $\mathcal{V}_{ij}$ are the ij elements of $\mathcal{V}_{CKM}$.) Then, the three equations for the invariants of the real matrix $LL^\dagger$ (the trace, the determinant and the

sum of its 2×2 determinants) become:

$$|L_{33}|^2 + 2|L_{23}|^2 \simeq m_\tau^2$$

$$|L_{22}L_{33} - L_{23}^2|^2 \simeq m_\mu^2 m_\tau^2$$

$$\text{Det}[LL^\dagger] = m_e^2 m_\mu^2 m_\tau^2 . \tag{5}$$

We find it is convenient to work in terms of the dimensionless parameters $\tilde{m}_i = m_i/m_\tau$ ($i = e, \mu$), $\tilde{a} = a\tilde{m}_t, \tilde{b} = b\tilde{m}_b$, and the ratios $r_c = m_c/m_t, r_s = m_s/m_b, r_u = m_u/m_t, r_d = m_d/m_b$. Explicitly from the equations above in terms of these parameters, we obtain:

$$\tilde{L}_{33} = e^{i\alpha_1} = \tilde{a}\mathcal{V}_{33}^2 + \tilde{b}\, e^{-iz_3}$$

$$\tilde{\Delta}_{23} = \tilde{m}_\mu e^{i\alpha_2} = (\tilde{b}\, r_s + \tilde{a}\, r_c e^{iz_2})\tilde{L}_{33} + \tilde{a}\, \tilde{b}\, \mathcal{V}_{32}^2 e^{i(b_b - b_s)}$$

$$\tilde{\Delta} = \tilde{m}_e \tilde{m}_\mu e^{i\alpha_3} = \tilde{b}\, r_d\, \tilde{\Delta}_{23} - \tilde{a}^2\, \tilde{b}\, e^{i(a_t - b_d)}\Big(r_s(\mathcal{V}_{31}\mathcal{V}_{33})^2$$

$$+ 2r_c \mathcal{V}_{31}\mathcal{V}_{32}\mathcal{V}_{21}\mathcal{V}_{22} e^{i(z_2 - z_3)} + r_c \mathcal{V}_{31}^2 \mathcal{V}_{22}\mathcal{V}_{33} e^{iz_2}\Big) \tag{6}$$

Here we have kept only the leading terms, using $r_u, r_d \ll r_c, r_s \ll 1$. Moreover, note that only phase differences like $a_i - b_i, a_i - a_j$ can be determinated from Eq. (3); therefore, by multiplying with overall phases, we have written Eqs. (6) in terms of these differences (with the notation $z_i = a_i - b_i$).

The key to solving this system is to recognize that there is some tuning involved. Analyzing the first two equations leads to the conclusion that $\tilde{a}, \tilde{b} \simeq \mathcal{O}(10)$. Then the phase z_3 in the first equation should be close to π so that the two terms almost cancel each other. Similar cancellations happen in the second and the third equations, which require respectively that $b_b - b_s \simeq \pi$ and $b_d \simeq a_t$[1]. Also, in the third equation, neglecting the small electron mass on the left hand side results in:

$$\tilde{a}^2 \simeq \frac{r_d\, \tilde{m}_\mu}{r_s |\mathcal{V}_{31}\mathcal{V}_{33}|^2} . \tag{7}$$

For values of the parameters in the experimentally feasible region, this is consistent with the above estimate $\tilde{a} \simeq \mathcal{O}(10)$.

Analytically solving Eqs. (6) with the approximations discussed will provide solutions for the phases and parameters $\tilde{a}, \tilde{b}$ accurate to the 10% level. Using these first order results, one can compute and put the neglected terms back in Eqs. (5,6), which can be solved again, thus defining

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[1] Note that taking these phase differences to π or zero results in exactly the mass signs which the analysis in [10] found to work for the real masses case.

an iterative procedure which can be implemented numerically, and brings us arbitrarily close to the exact solution. We find that 5 to 10 iterations are usually sufficient to recover the μ and τ masses with better than 0.1% accuracy (m_e can be brought to a fixed value by multiplying with an overall coefficient).

We end this section with some comments on the range of input parameters (masses and phases) which allow for a solution to Eq. (3). As we discussed above, the phases are either close to π or to zero. This is required by the necessity to almost cancel two large terms in the right-hand side of Eqs. (6). One can see that the larger the absolute magnitude of these terms (for example $|\tilde{a}\mathcal{V}_{33}^2|$ and $\tilde{b}$ in the first equation), the more stringent are the constraints on the phases. The opposite is also true; the smaller the $\tilde{a}$ and $\tilde{b}$ parameters, the more the phases can deviate from π, and generally the easier it is to solve the system. This means that lower values of $\tilde{a}$, $\tilde{b}$ are preferred; from Eq. (7), this implies a preference for low values of the ratio m_d/m_s [2] (there is not much scope to vary $\tilde{m_\mu}$). It turns out that low values of m_s and large values of m_c can also help, since they lower the absolute magnitude of the larger term on the right-hand side of the equation for $\tilde{\Delta}_{23}$ in (6). Previous analysis found indeed that fitting for the lepton masses require a low value for m_s [13].

A. Low scale values and RGE running

As was discussed in the above section, the relation (3) implies some constraints on the quark masses (the lepton masses being taken as input). That is, not all values of quark masses consistent with the experimental results are also consistent with the model we use. Our purpose first is to identify these points in the parameter space defined by the experimentally alowed values for quark masses,

Let us then define what this parameter space is. Altought the relations in the previous section hold at GUT scale, one must necessarily start with the low energy values for our parameters. We choose to use as input the values of the quark masses and the CKM angles at the M_Z scale. Estimates of these quantities can be found for example in [22]. However, we consider some of their numbers rather too precise (for example, their error in estimating the masses of the s and c quarks are only 25%, respectively 15%, while the corresponding errors in PDG [23] are much larger). Therefore, in the interest of making the parameter space as large as possible, we use the

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[2] This also means higher values for $|\mathcal{V}_{31}|$ are preferred. Since $\mathcal{V}_{31} = s_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta}$, this implies a preference for values of the CKM phase δ close to π (as noted in [12]).

following values:

- for the second family: 70 MeV $< m_s(M_Z) <$ 95 MeV [3]; 650 MeV $< m_c(M_Z) <$ 850 MeV. With a running factor from M_Z to 2 GeV of around 1.7, these limits would translate to values at 2 GeV scale of: 120 MeV $\lesssim m_s(2 \text{ GeV}) \lesssim$ 160 MeV ; 1.1 GeV $\lesssim m_c(2 \text{ GeV}) \lesssim$ 1.44 GeV. Lattice estimations $(m_c/m_s)_{2GeV} \simeq 12$ [24] would indicate a value in the lower part of the range for m_s, and a upper part for m_c.

- for the light quarks: here generally the ratio of quark masses are more trustworthy than limits on the masses themselves; we therefore use use $17 < m_s/m_d < 23$ (as noted in the previous section, high values of this ratio are preferred), and $0.3 < m_u/m_d < 0.7$. We note here that m_u is a parameter which does not affect the results much.

- for the heavy quarks: 2.9 GeV $< m_b(M_Z) <$ 3.11 GeV, (or 4.23 GeV $< m_b(m_b) <$ 4.54 GeV) and for the pole top mass 171 GeV $< M_t <$ 181 GeV (the corresponding $\bar{MS}$ mass is evaluated using the three loops relation, and comes out about 10 GeV smaller).

- the CKM angles at M_Z scale:

$$s_{12} = 0.222 \pm 0.003 \ , \ \ s_{23} = 0.04 \pm 0.004 \ , \ \ s_{13} = 0.0035 \pm 0.0015.$$

For the gauge coupling constants we take the following values at M_Z scale: $\alpha_1(M_Z) = 1/58.97, \alpha_2(M_Z) = 1/29.61, \alpha_3(M_Z) = 0.118$. With these values at low scale one can get unification of coupling constants at the scale $M_{GUT} \sim 10^{16}$. The exact value of M_{GUT}, as well as the values of the fermions Yukawas at the unification scale, will depend also on the supersymmetry breaking scale (M_{SUSY}) and $\tan\beta$, the ratio between the up-type and down-type SUSY Higgs VEVs. We generally consider values of M_{SUSY} between 200 GeV and 1 TeV, and $\tan\beta$ between 5 and 60.

Having chosen specific values of the parameters described above, we then run the fermion Yukawa coupling and the quark sector mixing angles, first from M_Z to M_{SUSY}, using two-loop Standard Model renormalization group equations; then we run from SUSY scale to the GUT scale using two loop[4] SUSY RGEs [26]. After computing the neutrino mass matrix at GUT scale, we

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[3] Note that the lower limit for $m_s(M_Z)$ is rather low compared with [22]; however, the value at 2 GeV scale is well within the limits cited in [23]. Lattice results also seem to favor smaller values of $m_s(2 \text{ GeV})$ [25].

[4] More precisely, we use the two-loop RGEs for the running of the gauge coupling constants and the third family fermions $(b, t$ and $\tau)$. To evaluate the light fermion masses, we use the one-loop equations for the ratios $m_{s,d}/m_b, m_{c,u}/m_t$ and $m_{\mu,e}/m_\tau$. This approximation is justified, since the leading two-loop effect on the fermion masses comes from the change in the values of gauge coupling constants at two-loop; however, the contributions due to the gauge terms are family-independent and will not affect these ratios.

run its elements back to Z scale [27, 28] and evaluate the resulting masses and mixing angles.

IV. NEUTRINO MASSES AND MIXINGS

In the present framework, there are two contributions to neutrino masses. First one has the canonical seesaw term:

$$(M_\nu)_{seesawI} = M_\nu^D M_R^{-1} M_\nu^D \qquad (8)$$

with $M_R = v_R Y_{126}$ and M_ν^D given by (1). However, the existence in this model of the $(\bar{\mathbf{10}}, \mathbf{3}, \mathbf{1})$ Higgs multiplet implies the possibility of a direct left-handed neutrino mass term when the $SU(2)_L$ Higgs triplet Δ_L from this acquires a VEV v_L (as it generally can be expected to happen). The neutrino mass contribution of such a term would be

$$(M_\nu)_{seesawII} = v_L Y_{126} = \lambda M_R \qquad (9)$$

where $v_L = \gamma v_{weak}^2 / v_R$ and γ is a factor depending on the specific form of the Higgs potential [21].

The scale of the canonical seesaw contribution Eq. (8) (which we call type–I seesaw in the following) to the left handed neutrino mass matrix is given by v_{weak}^2 / v_R. The contribution of the type–II seesaw factor (Eq. (9)) is of order $\gamma v_{weak}^2 / v_R$. One cannot know apriori how the factor γ compares with unity, therefore one cannot say which type of seesaw dominates (or if they are of the same order of magnitude). Therefore, in the following each case will be analyzed separately.

However, let us first review the current experimental data on the neutrino mixing angles and mass splittings. Latest analysis [29] sets the following 3σ bounds:

- from $\nu_\mu - \nu_\tau$ oscillations:

$$1.4 \times 10^{-3} \text{ eV}^2 \leq \Delta m_{23}^2 \leq 3.3 \times 10^{-3} \text{ eV}^2 \;\; ; \;\; 0.34 \leq \sin^2 \theta_{23} \leq 0.66 \; ;$$

  with the best fit for $\Delta m_{23}^2 = 2.2 \times 10^{-3} \text{ eV}^2$ and $\sin^2 \theta_{23} = 0.5$ (from atmospheric and K2K data).

- from $\nu_e - \nu_\mu$ oscillations:

$$7.3 \times 10^{-5} \text{ eV}^2 \leq \Delta m_{12}^2 \leq 9.1 \times 10^{-5} \text{ eV}^2 \;\; ; \;\; 0.23 \leq \sin^2 \theta_{12} \leq 0.37 \; ;$$

  with the best fit for $\Delta m_{12}^2 = 8.1 \times 10^{-5} \text{ eV}^2$ and $\sin^2 \theta_{12} = 0.29$ (from solar and KamLAND data). Note also that a previously acceptable region with a somewhat higher mass splitting $\Delta m_{12}^2 \simeq 1.4 \times 10^{-4} \text{ eV}^2$ (the LMA II solution [30]) is excluded now at about 4σ by the latest KamLAND data.

- finally, by using direct constraints from the CHOOZ reactor experiment as well as combined three-neutrino fitting of the atmospheric and solar oscillations, one can set the following upper limit on the θ_{13} mixing angle:

$$\sin^2 \theta_{13} \leq 0.022 \ .$$

The procedure we use in searching for a fit to neutrino sector parameters is as follows. First the low scale values of the quark and lepton masses and the CKM matrix angles and phase are chosen. (Generally we take a fixed value for m_b and m_t, while the other parameters are chosen randomly from a predefined range; however, m_b and m_t can also be chosen randomly). Next we pick a value for $\tan\beta$ and M_{SUSY}, and compute the quark-lepton sector quantities at GUT scale. Here we determine the relation between the lepton Yukawa couplings and quark Yukawa couplings, which amounts to determining the parameters a, b and phases a_i, b_i in Eq. (3). The phases combinations $z_i = a_i + b_i$ are chosen as input (that is, they are picked randomly), while a, b, and the remaining two phases are obtained by the procedure of fitting the lepton eigenvalues described in Section III. Finally, we scan over the parameters which appear in the neutrino sector (if the neutrino mass matrix is either of type–I seesaw or of type–II seesaw, there is only one phase σ; if both types appear, there will be two extra parameters, the relative magnitude and phase of the two contributions).

The rest of this section is devoted to a detailed analysis of the predictions of the $SO(10)$ minimal model for the neutrino sector, in type–I, type–II and mixed scenarios. (Due to its relative simplicity, we will start with the type–II case). However, let us first summarize our results. We find that in the type–II scenario, there is no good fit to the neutrino sector if the CKM phase is consistent with experimental measurements (around 60 deg). This is in agreement with previous analysis [12, 13]; however, our results are a bit more encouraging, in that that for $\delta_{CKM} = \pi/2$ we find reasonably good fits, which improve significantly with not very large increases in the CKM phase. We can obtain marginal fits for δ_{CKM} as low as 80 deg. More interesting are the results for the type–I case; here we can find good fits to the neutrino sector for values of δ_{CKM} as low as 50°, certainly consistent with experimental limits. As such fits have been not found before, one might consider this to be the main result of our paper. Also, we find that in the mixed case, there is possible to obtain a good neutrino sector fit in the case when the contributions coming from type–I and type–II are roughly equal in magnitude and of opposite phase.

A. Example of Type–I Seesaw Fit

We give here a representative example of a fit obtained in a type–I dominant case. This is obtained for $m_s(M_Z) = 0.07$ GeV, $m_c(M_Z) = 0.85$ GeV, $m_b(M_Z) = 3$ GeV, $M_t = 174$ GeV, $\tan \beta = 40$ and $M_{SUSY} = 500$ GeV. The values of the quark and lepton masses at GUT scale (in GeV) and the CKM angles are:

$$
\begin{aligned}
m_u &= 0.0006745 & m_c &= 0.3308 & m_t &= 97.335 \\
m_d &= 0.0009726 & m_s &= 0.02167 & m_b &= 1.1475 \\
m_e &= 0.000344 & m_\mu &= 0.0726 & m_\tau &= 1.350 \\
s_{12} &= 0.2248 & s_{23} &= 0.03278 & s_{13} &= 0.00216 \\
& & \delta_{CKM} &= 1.193 \; .
\end{aligned}
\tag{10}
$$

Here the masses are defined as $m_u = Y_u \sin \beta \; v_0$, $m_d = Y_d \cos \beta \; v_0$ where Y_u, Y_d are the corresponding Yukawa couplings, and $v_0 = 174$ GeV is the SM Higgs vacuum expectation value[5]. The values of GUT scale phases (in radians) and a, b parameters are given by:

$$
\begin{aligned}
a_u &= 0.881 & a_c &= 0.32678 & a_t &= 3.0382 \\
b_d &= 3.63235 & b_s &= 3.23784 & b_b &= 0. \\
a &= 0.08136 & b &= 5.9797 & \sigma &= 3.244 \; .
\end{aligned}
\tag{11}
$$

With these inputs, one can evaluate all mass matrices at GUT scale. In order to compute the neutrino mass matrix at M_Z scale, we use the running factors $r_{22} = 1.06, r_{23} = 1.03$, where

$$
r_{22} = \left(\frac{M_{\nu ij}}{M_{\nu 33}} \right)_{M_Z} \Big/ \left(\frac{M_{\nu ij}}{M_{\nu 33}} \right)_{M_{GUT}} \quad , \quad r_{23} = \left(\frac{M_{\nu i3}}{M_{\nu 33}} \right)_{M_Z} \Big/ \left(\frac{M_{\nu i3}}{M_{\nu 33}} \right)_{M_{GUT}} \quad ,
$$

with $i, j = 1, 2$. The elements of the neutrino matrix above are evaluated in a basis where the lepton mass matrix is diagonal.

One then obtains for the neutrino parameters at low scale:

$$
\Delta m_{23}^2 / \Delta m_{12}^2 \simeq 24 \; , \; \sin^2 \theta_{12} \simeq 0.27 \; , \; \sin^2 2\theta_{23} \simeq 0.90 \; , \; \sin^2 2\theta_{13} \simeq 0.08 \; .
$$

Note here that only the atmospheric angle is close to the experimental limit, the solar angle and the mass spliting ratio being close to the preferred values. The elements of the diagonal neutrino

---

[5] One can write Eqs. (3), (12) in terms of either the Yukawa couplings of the leptons and quarks, or their masses (that is, Yukawa couplings times running Higgs VEVs). In this paper we use the Yukawa couplings, but we multiply by the Higgs VEVs at the SUSY scale for simplicity of presentation. One can easily check then when going from one convention to the other, just the parameter a rescales, while b does not change.

mass matrix are

$$m_{\nu i} \simeq \{0.0021 \exp(0.11i) ,\ 0.0098 \exp(-3.06i) ,\ 0.048\}$$

in eV, with a normalization $\Delta m_{23}^2 = 2.2 \times 10^{-3} \text{eV}^2$. The phases of the first two masses are the Majorana phases (in radians). Moreover, the Dirac phase appearing in the MNS matrix is $\phi_D = -0.23 \text{rad}$, and one evaluates the effective neutrino mass for the neutrinoless double beta decay process to be

$$|\sum U_{ei}^2 m_{\nu i}| \simeq 0.009 \text{ eV} .$$

B. Type–II seesaw

Much of the recent work on the neutrino sector in the minimal $SO(10)$ has concentrated on the scenario when the type–II seesaw contribution to neutrino masses is dominant. The reason for the interest in this case is that, with:

$$M_\nu \sim M_R \sim M_l - M_d .$$

$b - \tau$ unification at the GUT scale, $m_b \simeq m_\tau$, naturally leads to a small value of $(M_\nu)_{33}$ and hence large mixing in the 2-3 sector [6]. However, while the general argument holds, it has been difficult (or impossible) to fit both large θ_{atm} and the hierarchy between the solar and atmospheric mass splittings at once. In this section we will try to show why this is so, and under which conditions this might be achievable.

We will use the same conventions as in section III (that is, we work in a basis where M_d is diagonal, and the parameters a, b and $(M_d)_{33} = m_b$ are real and positive). However, in the construction of the neutrino mass matrices there will be an extra phase besides those which were relevant for the quark-lepton mass matrices. This phase σ can be though as an overall phase of M_l. One then has:

$$
\begin{aligned}
M_R &= y(e^{i\sigma} M_l - M_d) \\
a M_\nu^D &= -(b e^{i\sigma} + 2) M_l e^{i\sigma} + 3 M_d .
\end{aligned}
\tag{12}
$$

Following the analysis in Sec. III one can write:

$$
\begin{aligned}
(M_l)_{22} &\simeq |b| \, m_s \, e^{ib_2} \\
(M_l)_{23} &\simeq a \, m_t \, e^{ia_3} \mathcal{V}_{32} \mathcal{V}_{33} \\
(M_l)_{33} &= a \, m_t \, e^{ia_3} + b \, m_b \simeq m_\tau e^{i\alpha} ,
\end{aligned}
\tag{13}
$$

with a_t close to π and $\alpha + b_s = \epsilon$ close to zero. Then, the neutrino mass matrix will be proportional to:

$$(M_\nu)_{[2,3]} \sim M_l - M_d{}^d e^{-i\sigma} \sim \begin{pmatrix} m_s e^{i(\epsilon-\alpha)}(b - e^{-i\sigma}) & m_{23} \\ m_{23} & m_\tau e^{i\alpha} - m_b e^{-i\sigma} \end{pmatrix} \tag{14}$$

Note also that $m_{23} = m_t\, e^{ia_t}\mathcal{V}_{32}\mathcal{V}_{33}$ is almost real positive, and due to the fact that $am_t \simeq bm_b$ and $m_b \mathcal{V}_{32} \simeq m_s$, the 22 and 23 elements in the neutrino mass matrix are roughly of the same order of magnitude (in practice, one get m_{23} somewhat larger than m_{22}). One then sees that if the phase σ is chosen such that the two terms in the 33 mass matrix element cancel each other (that is $\sigma \simeq -\alpha$) then there will be large mixing in the 2-3 sector, with:

$$\tan(\theta_\nu)_{23} \simeq \left| \frac{m_{23}}{m_{33}} \right| .$$

However, this is not the whole story. One needs also some hierarchy between the atmospheric and solar neutrino mass splittings:

$$\frac{\Delta m_{sol}^2}{\Delta m_{atm}^2} = r \lesssim \frac{1}{20}$$

(based on the experimental measurements of neutrino parameters reviewed in the previous section). In terms of the eigenvalues m_2, m_3 of the mass matrix (14), one then has:

$$\frac{m_2^2}{m_3^2} \simeq \left(\frac{|m_{22}m_{33} - m_{23}^2|}{|m_{22}^2| + |m_{33}^2| + 2|m_{23}|^2} \right)^2 \lesssim \frac{1}{20} . \tag{15}$$

In order for this to hold, one needs a cancellation between the $m_{22}m_{33}$ and m_{23}^2 terms in the numerator of the above fraction. This in turn imposes a constraint on the phases involved:

$$\phi = \mathrm{Arg}(m_\tau - m_b e^{-i(\sigma-\alpha)}) \simeq 0 . \tag{16}$$

More detailed analysis shows that it is not possible (or very difficult) to get $\tan(\theta_\nu)_{23}$ larger than 1 while satisfying the relation (15) between eigenvalues. However, this will create problems with the atmospheric mixing angle. The PMNS matrix is

$$U_{PMNS} = U_l^\dagger U_\nu$$

where U_l, U_ν are the matrices which diagonalize the lepton and neutrino mass matrices, respectively. Since the lepton mass matrix has a hierarchical form, the U_l matrix is close to unity, with $(U_l)_{i3} \simeq (0, x, 1)$, where $x = (m_{23}/m_{33})_l^*$. The atmospheric mixing angle will then be:

$$\tan\theta_{\mathrm{atm}} \simeq \left| \frac{\left(\frac{m_{23}}{m_{33}}\right)_\nu^* - \left(\frac{m_{23}}{m_{33}}\right)_l^*}{1 + \left(\frac{m_{23}}{m_{33}}\right)_\nu^* \left(\frac{m_{23}}{m_{33}}\right)_l} \right|$$
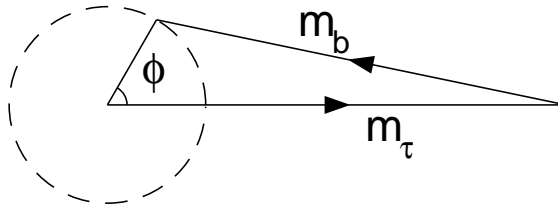
FIG. 1: Graphical representation of the relationship between the b and τ masses at the GUT scale and the magnitude and phase of the 33 element of the neutrino mass matrix.

where the l and ν lower indices make clear that we are discussing elements of the lepton and neutrino mass matrices. Note, however, that Eq. (16) implies that $(m_{33})_l$ and $(m_{33})_\nu$ have the same phase of m_τ; then, since $(m_{23})_l = (m_{23})_\nu$, the net effect of the rotation coming from the lepton sector is to reduce the 2-3 mixing angle. Practically, since $|x| \approx 0.2$, even if one has a value of $\tan(\theta_\nu)_{23}$ close to one from the neutrino mass matrix, $\tan\theta_{\mathrm{atm}}$ will become of order 0.7 after rotation in the lepton sector is taken into account.

This situation is represented graphically in Fig. 1. As discussed above, $\phi = 0$ corresponds to the case most favorable for getting the right solar-atmospheric mass splitting ratio, while $\phi = \pi$ corresponds to the case of maximal mixing angle. In practice this means that most reasonable fits are actually obtained when the angle ϕ is close to around $\pi/2$ (otherwise generally either the angle or the mass ratio are too small) [6]. Note however that $\phi = \pi$ (or any value greater than about $\pi/2$) would require that at GUT scale $m_b > m_\tau$. We may infer that in order to obtain a large mixing in the 2-3 sector one needs that at GUT scale m_b should be close to m_τ.

This in turn can be insured by requiring large $m_b(M_Z)$ and/or $\tan\beta$. For example, in Fig. 2 we present the results obtained for values $m_b(M_Z) = 3.11$ GeV, $m_t = 181$ GeV, $M_{SUSY} = 500$ GeV and $\tan\beta = 50$ (with these values, the ratio $r_{b/\tau} = m_b/m_\tau(M_{GUT}) \simeq 0.96$). Also, here we set the CKM phase $\delta = 90\,\mathrm{deg}$, and let the other quark sector parameters vary between the limits discussed in section IIIA. The left panel shows the maximum atmospheric/solar mass splitting ratio $R_{a/s}$ as a function of the atmosperic mixing angle $\theta_{atm} = \theta_{23}$. The three different lines correspond to

---

[6] Contributions from the phases in the lepton mass matrix can also improve the goodness of the fit (for example, if ϵ is significantly different from zero, or α different from π). However, this generally requires that the parameters $\tilde{a}, \tilde{b}$ have low values (as explained in section III). Hence we see that neutrino sector also prefers δ_{CKM} in the second quadrant and low m_s.
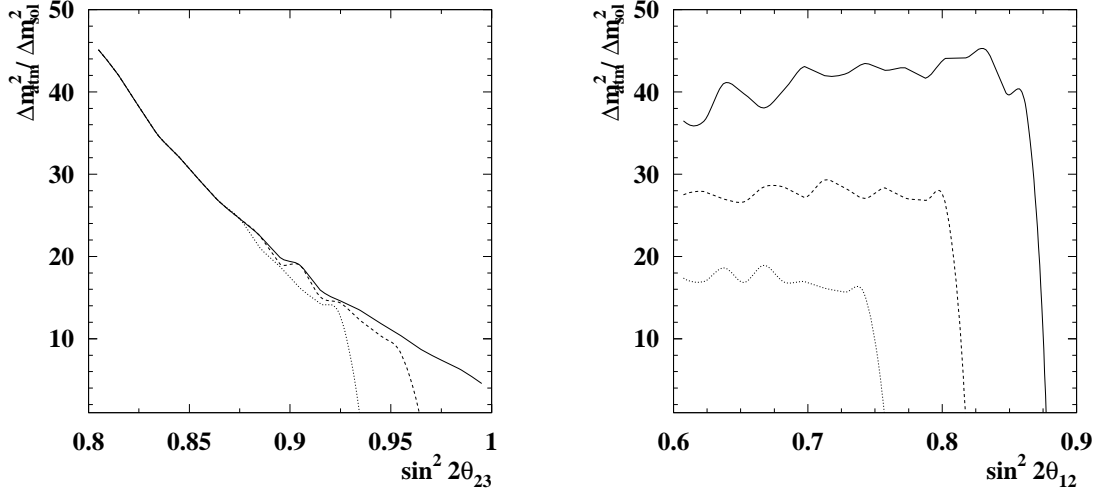
FIG. 2: Left: maximum of the atmospheric/solar mass splitting ratio $R_{a/s}$ as a function of the atmospheric mixing angle θ_{23}, with cuts on the solar angle $\sin^2 2\theta_{12} > 0.7$ (dotted line), 0.65 (dashed) and no cut (solid). Right: maximum of $R_{a/s}$ as a function of the solar mixing angle θ_{12}, with cuts on the atmospheric angle $\sin^2 2\theta_{23} > 0.9$ (dotted line), 0.85 (dashed) and 0.8 (solid).

different cuts on the solar mixing angle: $\sin^2 2\theta_{12} > 0.7$ (dotted), $\sin^2 2\theta_{12} > 0.65$ (dashed) and no cut (solid). One can observe here the correlation betewee large atmospheric mixing and small atmospheric-solar mass ratio.

Conversely, the right panel shows the maximum of the ratio $R_{a/s}$ as a function of the solar mixing angle $\theta_{sol} = \theta_{12}$. The three different lines correspond to cuts on the atmospheric mixing angle: $\sin^2 2\theta_{23} > 0.9$ (dotted), $\sin^2 2\theta_{23} > 0.85$ (dashed) and $\sin^2 2\theta_{23} > 0.8$ (solid line). We note here that the correlation between the solar angle and the mass ratio has the form of a step function (abrupt decrease in $R_{a/s}$ once $\sin^2 2\theta_{12}$ goes over a certain threshold), while there seems to be a close to linear correlation between the maximal solar and atmospheric angles.

It is interesting to consider how these results change if the paramenters $m_b(M_Z), m_t, M_{SUSY}$ and/or $\tan \beta$ are modified. One finds out that the neutrino sector results have a strong dependence on the parameter $\tan \beta$. For example, if one keeps the parameters used in Fig. 2 fixed but increases $\tan \beta$, one finds that the fit for the atmospheric angle - atmospheric/solar mass ratio improves to a certain amount. That can be traced to the fact that the ratio $r_{b/\tau}$ increases with $\tan \beta$. However, one also finds that the solar angle generally gets smaller. This happens because there is a correlation between the solar angle and the value of the s quark mass at GUT scale; namely θ_{12} increases with $r_{s/b} = m_s/m_b(M_{GUT})$. On the other hand, the ratio $r_{s/b}$ decreases with increasing $\tan \beta$.
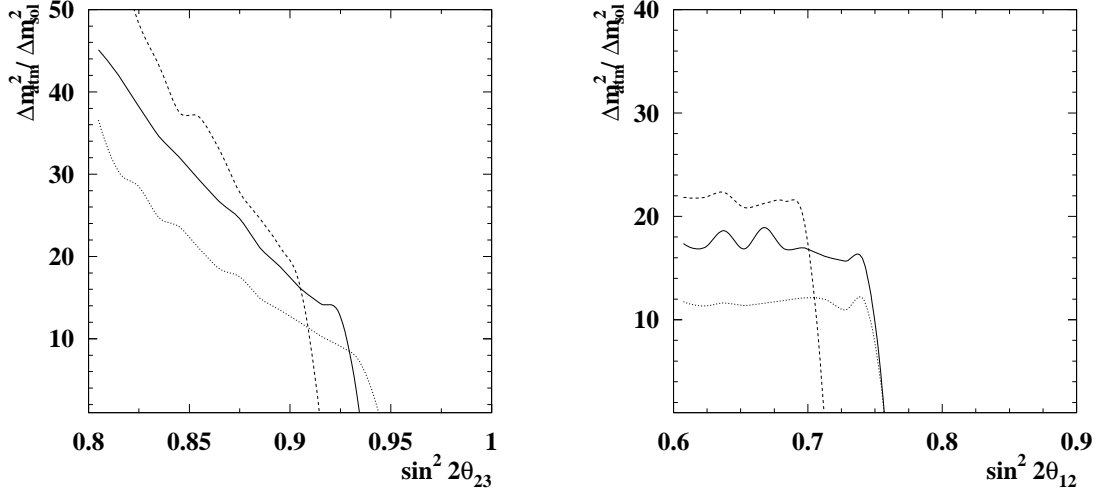
FIG. 3: Maximum of the ratio $R_{a/s}$ as a function of the atmospheric mixing angle θ_{23} (left) and solar angle (right), for $\tan\beta = 40$ (dotted line), 50 (solid) and 55 (dashed). Additional cuts are $\sin^2 2\theta_{12} > 0.7$ (left) and $\sin^2 2\theta_{23} > 0.9$ (right).

Fig. 3 exemplifies this behaviour. The three lines correspond to the maximum value for the mass splitting ratio $R_{a/s}$, at values of $\tan\beta = 40$ (dotted), $\tan\beta = 50$ (solid) and $\tan\beta = 55$ (dashed line). A cut on the solar angle $\sin^2 2\theta_{12} > 0.7$ is also imposed in the left panel, and a cut on the atmospheric angle $\sin^2 2\theta_{12} > 0.9$ in the right panel. One can see that at larger values for $\tan\beta$ one might potentially get better fits for atmospheric angle and the atmospheric/solar mass ratio; however, the constraint on the solar angle becomes more restrictive.

Smaller variations of the neutrino sector results will follow modification of the parameters $m_b(M_Z), m_t$ and M_{SUSY}. However, these variations follow the same pattern as above: that is, an improvement in the fit for the atmospheric angle due to the increase of the ratio $r_{b/\tau}$ (which can be due to an increase in m_t, or a decrease in M_{SUSY}) coincide with a worsening of the fit for the solar angle. As a consequence, the results presented in Figs. 2, 3 can be improved only marginally. Scanning over a range of parameter space $2.9\text{GeV} < m_b(M_Z) < 3.11\text{GeV}$, $174\text{GeV} < m_t < 181\text{GeV}$, $500\text{GeV} < M_{SUSY} < 1\text{TeV}$ and $10 < \tan\beta < 60$ [7], we find the best fit to the neutrino sector to be $\sin^2 2\theta_{23} \simeq 0.88$, $\sin^2 2\theta_{12} \simeq 0.74$ and the atmospheric/solar mass splitting ratio $R_{a/s} \simeq 24$. We note that although these numbers provide a somewhat marginal fit to the

---

[7] In practice we find that the best results are obtained for large $m_b(M_Z)$, large m_t and large $\tan\beta$ (such that $r_{b/\tau}$ is between 0.96 and 1).
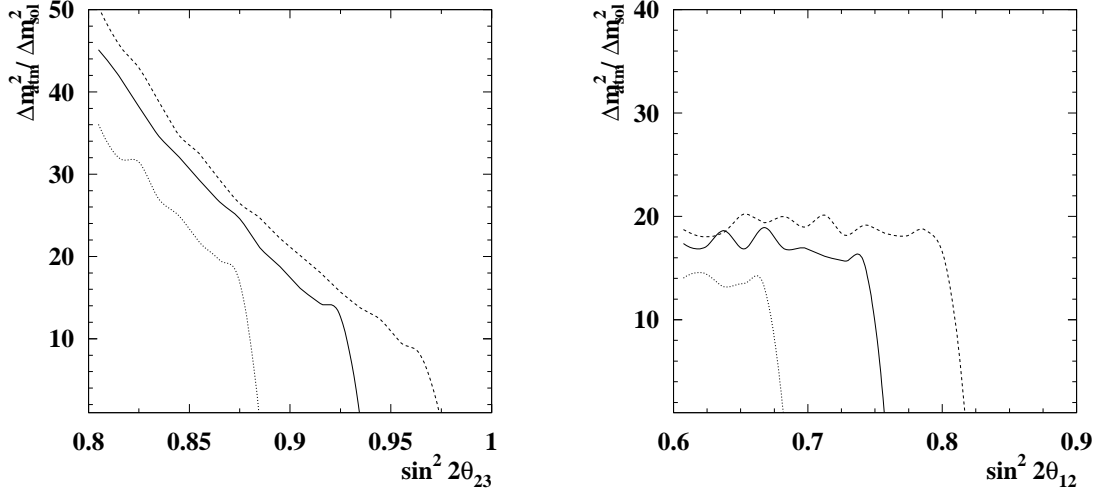
FIG. 4: Maximum of the ratio $R_{a/s}$ as a function of the atmospheric mixing angle θ_{23} (left) and solar angle (right), for $\delta_{CKM} = 80^o$ (dotted line), 90° (solid) and 100° (dashed). Additional cuts are $\sin^2 2\theta_{12} > 0.7$ (left) and $\sin^2 2\theta_{23} > 0.9$ (right).

experimental results (the mixing angles are close to the exclusion limit, while the value for mass ratio is central) they are still allowed.

However, the results discussed above were obtained for a value of the CKM phase $\delta = 90\,\mathrm{deg}$ which is too large compared with the measured value ($\delta_{CKM} = 59 \pm 13\,\mathrm{deg}$ from PDG [23]). As argued in section III (and indeed noted by previous analysis) there is a strong dependence on the goodness of the fit on the value of δ_{CKM}, with larger values giving better fits. We show this dependence in Fig. 4. The parameters are the same as in Fig 2 ($m_b(M_Z) = 3.11$ GeV, $m_t = 181$GeV, $M_{SUSY} = 500$ GeV and $\tan\beta = 50$), but the three lines correspond to different values for δ_{CKM}: 80° (dotted line), 90° (solid) and 100° (dashed line). One can notice a rapid deterioration in the goodness of the fit with decreased δ_{CKM}. Thus, for $\delta_{CKM} = 80\,\mathrm{deg}$, the best fit to neutrino sector we find (after scanning over the SUSY parameter space) is $\sin^2 2\theta_{23} \simeq 0.88$, $\sin^2 2\theta_{12} \simeq 0.7$ and $R_{a/s} \simeq 18$.

For purposes of illustration, we give a fit obtained for a type–II dominant case for $\delta_{CKM} = 80\,\mathrm{deg}$, $m_b(M_Z) = 3.11$ GeV, $M_t = 181$ GeV, $\tan\beta = 55$ and $M_{SUSY} = 1$ TeV. The s, c quark masses at low scale are $m_s(M_Z) = 0.074$ GeV, $m_c(M_Z) = 0.83$ GeV. Then the values of the quark and lepton masses at GUT scale are (in GeV):

$$
\begin{aligned}
m_u &= 0.0008185 & m_c &= 0.3772 & m_t &= 139.876 \\
m_d &= 0.0015588 & m_s &= 0.03554 & m_b &= 2.3547 \\
m_e &= 0.000525 & m_\mu &= 0.1107 & m_\tau &= 2.420 \\
s_{12} &= 0.225 & s_{23} &= 0.0297 & s_{13} &= 0.00384 \\
& & \delta_{CKM} &= 1.4 \ .
\end{aligned}
\tag{17}
$$

The values of GUT scale phases (in radians) and a, b parameters are given by:

$$
\begin{aligned}
a_u &= -0.4689 & a_c &= -1.0869 & a_t &= 3.0928 \\
b_d &= 2.6063 & b_s &= 2.2916 & b_b &= 0. \\
a &= 0.09093 & b &= 4.423 & \sigma &= 3.577 \ .
\end{aligned}
\tag{18}
$$

The running factors for the neutrino mass matrix are $r_{22} = 1.09, r_{23} = 1.18$.

One then obtains for the neutrino parameters at low scale:

$$
\Delta m_{23}^2 / \Delta m_{12}^2 \simeq 18 \ , \ \sin^2 2\theta_{12} \simeq 0.7 \ , \ \sin^2 2\theta_{23} \simeq 0.88 \ , \ \sin^2 2\theta_{13} \simeq 0.094 \ .
$$

The elements of the diagonal neutrino mass matrix (masses and Majorana phases) are

$$
m_{\nu i} \simeq \{0.0016 \exp(0.27i) \ , \ 0.011 \exp(-2.86i) \ , \ 0.048\}
$$

in eV.

The Dirac phase appearing in the MNS matrix is $\phi_D = -0.007$rad, and one evaluates the effective neutrino mass for the neutrinoless double beta decay process to be

$$
|\sum U_{ei}^2 m_{\nu i}| \simeq 0.01 \text{ eV} \ .
$$

C. Type–I seesaw

The fact that in type–II seesaw one can obtain large mixing in the 23 sector is due to a lucky coincidence: the type–II neutrino mass matrix being written as a sum of two hierarchical matrices (M_l and M_d), the most natural form for the neutrino mass matrix is also hierarchical. However, since 33 elements of both matrices M_l and M_d are roughly of the same magnitude, by choosing the relative phase between the two to be close to π, one can get a neutrino mass matrix of the form suited to explain large mixing in the 2-3 sector.

The question arises then if such a coincidence happens for the type–I seesaw neutrino mass matrix. To see that, let's write the Dirac neutrino mass matrix in the following form:

$$M_\nu^D = \frac{be^{i\sigma} + 2}{a} \left[\tilde{M}_R + \frac{b - e^{-i\sigma}}{be^{i\sigma} + 2} M_d \right] \sim \tilde{M}_R + \tilde{M}_d$$

where $\tilde{M}_R$ is the scaled right-handed neutrino mass matrix $\tilde{M}_R = M_l - M_d e^{i\sigma}$ and $\tilde{M}_d$ is a rescaled down-type quark diagonal mass matrix (the scaling factor in this later case $\zeta = (b - e^{-i\sigma})/(be^{i\sigma} + 2)$ is close to unity, since b is roughly of order 10). Then the type–I seesaw neutrino mass matrix would be:

$$M_{\nu I} = M_\nu^D M_R^{-1} M_\nu^D \sim \tilde{M}_R + 2\tilde{M}_d + \tilde{M}_d \tilde{M}_R^{-1} \tilde{M}_d \ . \tag{19}$$

Now, for most values of the phase σ, $\tilde{M}_R$ is hierarchical, therefore so is $\tilde{M}_R^{-1}$, therefore the type–I neutrino mass matrix is the sum of three hierarchical matrices ($\tilde{M}_d$ being diagonal). So it is not surprising that for most values of the phase σ $M_{\nu I}$ is also hierarchical. What is remarkable is that there are some values of σ for which the type–I seesaw mass matrix has a large mixing in the 2-3 sector, and moreover, this happens for the same values of σ as in the case when the type–II mass matrix is non-hierarchical (that is, σ close to π).

In order to see this let us consider the magnitude and the phase of the 33 elements (the largest ones) in the three terms on the right-hand side of Eq. (19). If σ is not close to π, the magnitude of $\tilde{M}_{R33}$ is of order m_b, with varying phase (ϕ in Fig. 1 in the first quadrant); the magnitude of $(\tilde{M}_d)_{33}$ is also of order m_b, and the phase $\sim -\sigma$. For the last term, we make use of the fact that $\tilde{M}_R$ being hierarchical, $(\tilde{M}_R^{-1})_{33} \simeq 1/(\tilde{M}_R)_{33} \simeq 1/m_b$; then $\tilde{M}_d \tilde{M}_R^{-1} \tilde{M}_d \simeq m_b$, with a phase close to -2σ. We see then that for most values of σ $(M_{\nu I})_{33}$ is of order m_b, while the off-diagonal elements are small. However, for $\sigma \sim \pi$, the cancellation in the 33 element of $\tilde{M}_R$ is matched by a cancellation between the 33 elements of the $2\tilde{M}_d$ and $\tilde{M}_d \tilde{M}_R^{-1} \tilde{M}_d$ terms from Eq. (19) (since the relative phase between these is also σ), thus leading to a non-hierarchical form for the type–I seesaw neutrino mass matrix.

The fine-tuning between different contributions to the neutrino mass matrix is thus a little bit more involved in the type–I seesaw case compared to the type–II seesaw, but it can still lead to large mixing in the 2-3 sector. Moreover, since the correlations between the input parameters and the neutrino mass matrix elements are not so strong, most of the constraints discussed in the above section do not hold (for example, m_b does not have to be necessarily very close to m_τ). This may lead one to believe that it is possible to obtain a better fit for the neutrino sector in type–I models, and we found that in fact this is the case.
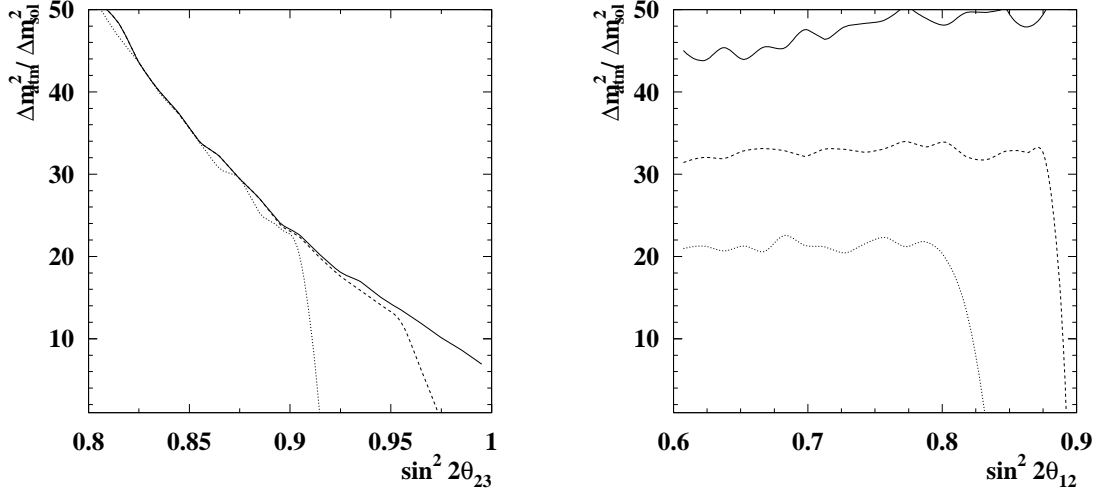
FIG. 5: Left: maximum of the atmospheric/solar mass splitting ratio $R_{a/s}$ as a function of the atmospheric mixing angle θ_{23}, with cuts on the solar angle $\sin^2 2\theta_{12} > 0.8$ (dotted line), 0.7 (dashed) and no cut (solid). Right: maximum of $R_{a/s}$ as a function of the solar mixing angle θ_{12}, with cuts on the atmospheric angle $\sin^2 2\theta_{23} > 0.9$ (dotted line), 0.85 (dashed) and 0.8 (solid).

For example, we show in Fig 5(left) the maximum atmospheric/solar mass splitting ratio $R_{a/s}$ as a function of the atmosperic mixing angle $\theta_{atm} = \theta_{23}$ - with cuts on the solar mixing angle: $\sin^2 2\theta_{12} > 0.8$ (dotted), $\sin^2 2\theta_{12} > 0.7$ (dashed) and no cut (solid). In the left panel we show maximum $R_{a/s}$ as a function of the solar angle for $\sin^2 2\theta_{23} > 0.9$ (dotted), $\sin^2 2\theta_{23} > 0.85$ (dashed) and $\sin^2 2\theta_{23} > 0.8$ (solid line). This figure is obtained for values $m_b(M_Z) = 3.0$ GeV, $m_t = 174$GeV, $M_{SUSY} = 500$GeV and $\tan\beta = 40$, while the CKM phase is allowed to vary between 60 and 70 deg. We see that it is possible to obtain a large atmosperic/solar mass splitting ratio for values of the atmospheric and solar mixings consisted with experimental constraints.

How do these results change if we modify the SUSY parameters $\tan\beta$ and M_{SUSY}, and/or the M_Z scale masses? We find that one can get good fits for values of the parameter $\eta_{b/\tau} = m_b/m_\tau(M_{GUT})$ between 0.83 and 0.9. For values of $m_b(M_Z) = 3.0$ GeV, $m_t = 174$ GeV, $M_{SUSY} = 500$ GeV, this means that $\tan\beta$ can vary between 10 and 55. However, if one increases m_b or m_t, generally $\eta_{b/\tau}$ will increase (there is also a slight decrease with increasing M_{SUSY}, but less pronounced). Thus, for $m_b(M_Z) = 3.11$ GeV, $m_t = 174$ GeV, one has to take $\tan\beta$ roughly between 10 and 45 in order to get a good neutrino result. Fig. 6 illustrates this behavior: the three lines correspond to values $\tan\beta = 20$ (solid), 45 (dashed) and 55 (dotted), for $m_b(M_Z) = 3.11$ GeV, $m_t = 174$ GeV and $M_{SUSY} = 500$ GeV the corresponding values for $\eta_{b/\tau}$ are 0.86, 0.9 and
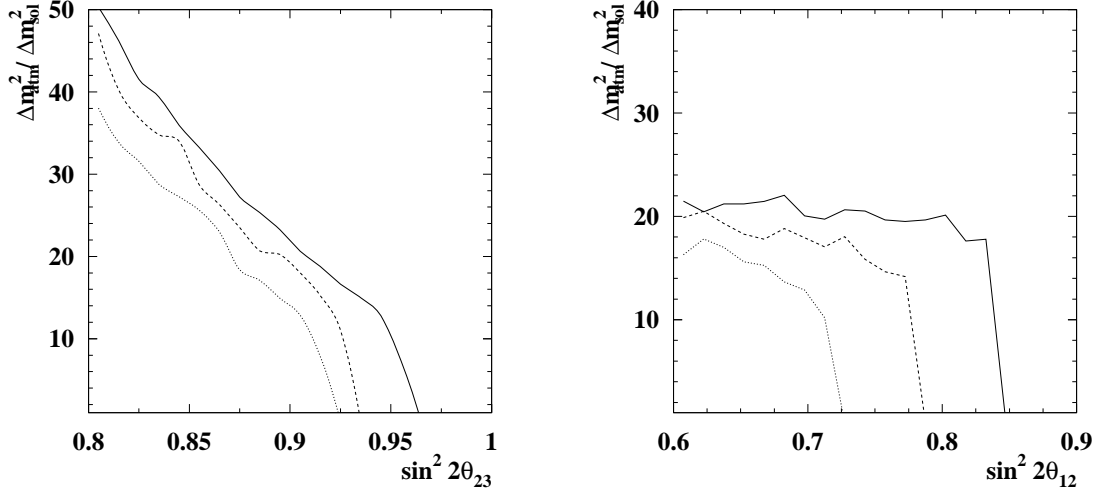
FIG. 6: Maximum of the ratio $R_{a/s}$ as a function of the atmospheric mixing angle θ_{23} (left) and solar angle θ_{12} (right), for $\tan\beta = 20$ (solid line), 45 (dashed) and 55 (dotted). Additional cuts are $\sin^2 2\theta_{12} > 0.7$ (left) and $\sin^2 2\theta_{23} > 0.9$ (right).

0.96. As noted above, the fit worsens dramatically for $\tan\beta > 45$. δ_{CKM} is taken between 60 and 70 degrees here also.

Finally, the θ_{13} mixing angle is found to lie in a range $0.06 \lesssim \sin^2 2\theta_{13} \lesssim 0.11$, with a preferred value ~ 0.085 . We show in Fig. 7 the distribution of values for the θ_{13} mixing angle and the Dirac phase in the neutrino mixing matrix δ_N (in radians) obtained for a scan of parameter space with $2.95 \text{ GeV} \lesssim m_b(M_Z) \lesssim 3.05 \text{ GeV}$, $172 \text{ GeV} \lesssim m_t \lesssim 176 \text{ GeV}$, $500 \text{ GeV} \lesssim M_{SUSY} \lesssim 750 \text{ GeV}$, and $0.83 \lesssim \eta_{b/\tau} \lesssim 0.9$. The cuts on the other neutrino sector parameters are $\sin^2 2\theta_{23} > 0.88$ (for all lines), $\sin^2 2\theta_{12} > 0.7$ and $R_{a/s} > 18$ (solid line), $\sin^2 2\theta_{12} > 0.75$ and $R_{a/s} > 18$ (dashed line), $\sin^2 2\theta_{12} > 0.7$ and $R_{a/s} > 20$ (dotted line). Note that the preferred value for θ_{13} is close to the experimental limit.

D.   Mixed type

We end by considering the scenario when both types of seesaw are present, and give a non-negligible contribution. One can envision two different situations: first when one contribution dominates, and the other can be treated as a small perturbation, and second when the two contributions are of roughly the same order of magnitude.

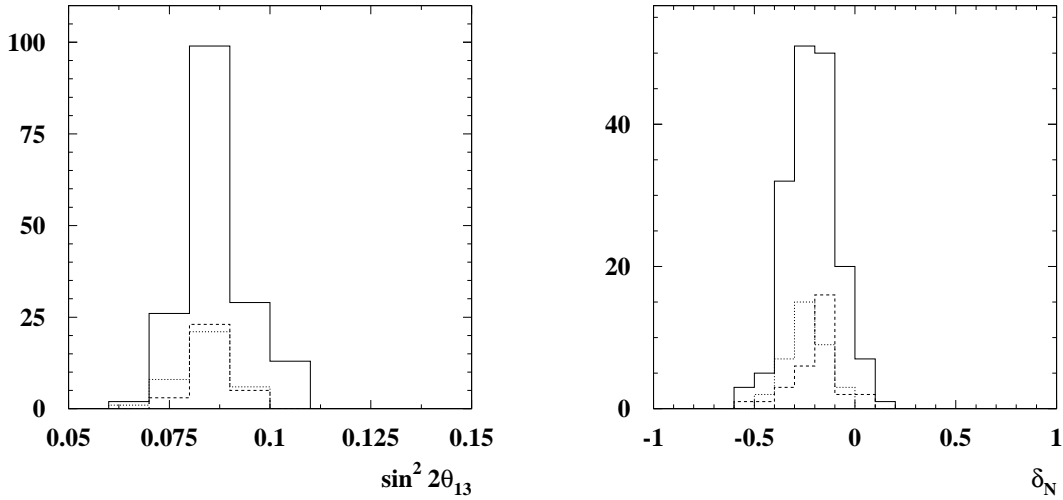Let us first discuss the first case. It is obvious that the fits one obtains when considering

FIG. 7: Distribution of values for the θ_{13} mixing angle (left) and the Dirac phase in the neutrino mixing matrix δ_N (right) obtained after a scan of the parameter space. The three lines correspond to different cuts on the other neutrino sector paramenters (see text for details).

the 'pure' case (either type–I or type–II) can only improve. Indeed, by adding the other type contribution, we introduce another two free parameters: the relative magnitude and phase of the subdominant contribution relative to the dominant one. One can then obtain the results for the 'pure' case by allowing the relative magnitude go to zero.

Thus there appears an interesting question. Is it possible to improve the fit for the type–II scenario to a level which is compatible with experimental data? Unfortunately, the answer seems to be no. Our numerical simulations show that by adding type–I contribution as a small perturbation, one obtains a small improvement over the 'pure' type–II case, but not a significant one.

This might seems counterintuitive at first glance. It would be reasonable to assume that if one takes the best result for a type–II fit and then adds another small quantity parametrized in terms of a free magnitude and phase, one should be able to obtain a somewhat better fit. The reason why this does not happen is that the type–I seesaw perturbation will modify in first order only the 33 neutrino matrix element[8]. However, as we saw from our analysis of type–II case, the goodness of the fit is determined mainly by the 22 and 23 matrix elements. That is, in type–II

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[8] It is easy to see that in the sum of last two terms on the right-hand side of Eq. (19) the 33 element is dominant if the type–II 23 mixing is close to maximal.

case, one generally has the freedom to easily modify the 33 element (by changing the phase σ or the ratio m_b/m_τ) so in a sense for a best fit m_{33} already has its optimal value, and adding a small perturbation to it does not buy any additional freedom. At most what one can achieve is to take some fit which is not optimal and make it better by adding the type–I perturbation.

Let us now consider the case when the contributions from the two types of seesaw are of comparable magnitude. In this case, the neutrino mass matrix can be thought of as having the form displayed in Eq. (19), with the first term $\tilde{M}_R$ (proportional to the type–II seesaw contribution) enhanced or diminished, depending on the relative phase between the two contributions. We find that if the type–II term is close to zero (that is, both contributions have about the same magnitude and the relative phase is close to π) then we can again get good fits for the neutrino sector. In particular, in this case we can obtain truly maximal mixing in the atmospheric sector ($\sin^2 2\theta_{23} \simeq 1$) with large enough mass splitting, and large enough solar angle. Also, obtaining these results does not require that we use a specific range of (low scale) parameters - like large $\tan\beta$, or large m_b. Generally, as long as one can fit for the lepton-quark sector, one can obtain a fit for the neutrino sector, too.

Here is a particular fit obtained in a mixed case. We took $\delta_{CKM} = 60\,\text{deg}$, $m_b(M_Z) = 3.0$ GeV, $M_t = 181$ GeV, $\tan\beta = 45$ and $M_{SUSY} = 500$ GeV. The s, c quark masses at low scale are $m_s(M_Z) = 0.071$ GeV, $m_c(M_Z) = 0.79$ GeV. Then the values of the quark and lepton masses at GUT scale are (in GeV):

$$
\begin{aligned}
m_u &= 0.0009440 & m_c &= 0.3958 & m_t &= 143.23 \\
m_d &= 0.001451 & m_s &= 0.02991 & m_b &= 1.6903 \\
m_e &= 0.00047 & m_\mu &= 0.0992 & m_\tau &= 1.910 \\
s_{12} &= 0.220 & s_{23} &= 0.0320 & s_{13} &= 0.00249 \\
& & \delta_{CKM} &= 1.05 \; .
\end{aligned}
\tag{20}
$$

The values of GUT scale phases (in radians) and a, b parameters are given by:

$$
\begin{aligned}
a_u &= -1.2527 & a_c &= -0.39256 & a_t &= 3.07385 \\
b_d &= 3.0896 & b_s &= -3.1367 & b_b &= 0. \\
a &= 0.089727 & b &= 6.6365 & \sigma &= 3.1505 \; .
\end{aligned}
\tag{21}
$$

The running factors for the neutrino mass matrix are $r_{22} = 1.04, r_{23} = 1.09$. We add the two contributions:

$$
M_\nu \; \sim \; M_{\nu II} \frac{(M_{\nu I})_{33}}{(M_{\nu II})_{33}} \; + \; M_{\nu I}\, r e^{i\phi} \; ,
$$

with relative magnitude $r = 0.9813$ and phase $\phi = 3.200$rad.

One then obtains for the neutrino parameters at low scale:

$$\Delta m_{23}^2/\Delta m_{12}^2 \simeq 28 \ , \ \sin^2 2\theta_{12} \simeq 0.85 \ , \ \sin^2 2\theta_{23} \simeq 0.98 \ , \ \sin^2 2\theta_{13} \simeq 0.05 \ .$$

The elements of the diagonal neutrino mass matrix (masses in eV and Majorana phases) are

$$m_{\nu i} \ \simeq \ \{0.0028 \exp(0.026i) \ , \ 0.0093 \exp(-3.07i) \ , \ 0.0478\} \ .$$

The Dirac phase appearing in the MNS matrix is $\phi_D = -0.3$rad, and one evaluates the effective neutrino mass for the neutrinoless double beta decay process to be

$$|\sum U_{ei}^2 m_{\nu i}| \ \simeq \ 0.009 \ \text{eV} \ .$$

V.   CONCLUSIONS

An $SO(10)$ model with only one **10** and one $\overline{\textbf{126}}$ Higgs multiplets coupling to fermions provides an appealing candidate for a unified theory at large scales. The number of free parameters in such a model is smaller than the number of free parameters in the Standard Model, thus giving the theory some predictive power. In particular, there appears the possibility that large mixing in the neutrino sector can be understood as a consequence of $b - \tau$ unification at GUT scale.

We revisit here the analysis of the minimal $SO(10)$ model and its implications for the neutrino sector. Our work differs from previous works in this area in several aspects. First, we consider the most general formulation of the model, with all the CP phases taken into consideration. Second, we use a new method for fitting the GUT scale parameters (Yukawa couplings) to the low scale masses and mixing angles in the quark and lepton sector. The running of the Yukawa couplings from low scale to unification scale is also taken into account, as well as dependence upon SUSY parameters like $\tan\beta$ and M_{SUSY}. We also analyze all possible cases for the neutrino mass generation, namely, when either the type–I or type–II seesaw mechanism dominates, or the case when both contributions are roughly of the same magnitude.

Our results are as follows. For the type–II seesaw case, we find that the requirement for close to maximal mixing in the 2-3 sector, together with the large hierarchy between atmospheric and solar mass splittings, pushes the CKM phase to large values (reasonably good fits can be found for $\delta_{CKM} > 100^o$). This is in agreement with previous results. Interestingly, we also find the requirement that the solar mixing angle is large imposes significant constraints. Better fits can be

found at large tanβ, and large values of $m_b(M_Z)$; however, the results obtained for values of δ_{CKM} in the first quadrant are at most marginally in agreement with experimental data.

More interesting is the case when the type–I seesaw mechanism dominates. Here, contrary to previous analyses, we find that it is possible to obtain good fits for the neutrino sector for values of δ_{CKM} as low as 50°, which includes the range consistent with experimental results. Actually, unlike the type–II case, it seems that the goodness of the fit is not very dependent on δ_{CKM}; the relevant parameters seem to be the values of m_c and m_s at M_Z scale, with preference for larger values for m_c and lower values for m_s. The solar angle also imposes significant constraints on the parameter space, but in this case it is possible to obtain good fits for a larger range of tanβ, $m_b(M_Z)$ and M_t.

It is also interesting to note that, for values of the phase σ appearing in the neutrino sector close to π, the type–I and type–II neutrino mass matrices are roughly proportional. This means that if the type–I mass matrix is non-hierarchical (thus leading to large mixing in the 2-3 sector), so will the type–II mass matrix be. Hence, if one considers the case when both contributions from type–I and type–II must be taken into account, one sees that the dominant contribution will determine the type of the fit. However, a special case is when the two contributions are roughly of the same magnitude, and their relative phase is close to π. In this case they cancel each other, and the resulting neutrino mass matrix is generally non-hierarchical. One can obtain very good fits for the neutrino sector in this case, too.

Acknowledgments

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